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*Published in:*  
Proceedings of the 2013 European Control Conference

*Publication date:*  
2013

[Link back to DTU Orbit](#)

*Citation (APA):*  
Tabatabaeipour, M., & Stoustrup, J. (2013). Set-membership state estimation for discrete time piecewise affine systems using zonotopes. In *Proceedings of the 2013 European Control Conference* (pp. 3143-3148). IEEE.

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# Set-membership state estimation for discrete time piecewise affine systems using zonotopes

S. Mojtaba Tabatabaeipour<sup>1</sup> and Jakob Stoustrup<sup>2</sup>

**Abstract**—This paper presents a method for guaranteed state estimation of discrete time piecewise affine systems with unknown but bounded noise and disturbance. Using zonotopic set representations, the proposed method computes the set of states that are consistent with the model, observation, and bounds on the noise and disturbance such that the real state of the system is guaranteed to lie in this set. Because in piecewise affine systems, the state space is partitioned into a number of polyhedral sets, at each iteration the intersection of the zonotopes containing a set-valued estimation of the states with each of the polyhedral partitions must be computed. We use an analytic method to compute the intersection as a zonotope and minimize the size of the intersection. A numerical example is provided to illuminate the algorithm.

## I. INTRODUCTION

In the context of control theory, knowing the states of the system is crucial to find the solution to many control problems. However, in practice all states of a system are not directly measurable. Therefore, it is very important to have an estimate of the state of the system. Amongst the approaches that are proposed for state estimation in the literature are stochastic methods, the  $H_\infty$  approaches, and set-membership approaches.

The stochastic approaches (Kalman filter theory) [13], propose a recursive method for computing a posteriori distribution of the state of the system by minimizing the error variance of estimates of the state. An important assumption in the Kalman filter method is that all the error terms and measurements have a known (usually Gaussian) distribution. This assumption about the statistical properties of the uncertainties are in many cases difficult to validate. In the  $H_\infty$  approaches, the measurement noise and disturbances are assumed to be arbitrary but with a bounded energy. Then, an optimal  $H_\infty$  filter with an  $H_\infty$  performance criterion is designed [14].

In set-membership approaches, noise, disturbance, and uncertainties are assumed to be unknown but bounded. Instead of a point-wise estimation of the states, a set-valued estimation of them is provided. A recursive filtering method is proposed to compute a compact set that is guaranteed to contain the set of states that are consistent with the measurement, the model of the system, and the bounded uncertainties.

To implement the algorithm, a particular set representation must be used since the exact computation of these sets is

extremely difficult. In the literature, different representations are proposed which include ellipsoids, polyhedrons, parallelotopes, intervals, or zonotopes, see [1], [16] and references therein. The specific representation must be efficient with regards to the operations that must be implemented in the algorithm. Amongst the first methods that were proposed for set-membership state estimation is [19] where a bounding ellipsoid which always contains the true state is computed. Ellipsoidal sets were later used in [11], for estimation and control. The advantage of ellipsoidal sets is their simplicity, but the problem with them is that ellipsoids are not closed under the Minkowski sum and intersection. Therefore, the sum and intersection of two ellipsoids must be over-approximated as an ellipsoids which results in a rather conservative solution. To obtain a better accuracy in the state estimation using polyhedral set were proposed by [10]. The advantage of the polyhedral sets is their accuracy. They are closed under the linear transformation, Minkowski sum, intersection, and convex hull computation. The drawback of the polyhedral sets is their computational complexity. Minkowski sum and convex hull computation for polytopes are in general restricted to systems with a maximum of 4-6 states. To address the problem of computational complexity using polyhedral sets, an approach based on minimum-volume bounding parallelotopes was presented in [4] and later in [3] for set-membership identification.

A zonotopes is a Minkowski sum of a number of line segments. Using zonotopes for worst case state estimation and simulation of uncertain systems was proposed in [15]. In [5] a set-membership method for state estimation using zonotopes is proposed. In [1] minimum-volume zonotopes are used for guaranteed state estimation of discrete-time non-linear systems. This method is later used for fault detection [16]. In the parameter domain, zonotopes are used in [2] to computed a set-valued estimation of the parameters of the system with the aim of system identification.

Set-membership state estimation methods has attracted a growing attention in the area of fault detection for robust fault detection when noise and uncertainties are explicitly taken into account [16], [6], [8]. Authors in [20], [21], use set-membership fault detection for fault detection in benchmark wind turbine using polyhedral sets and zonotopes respectively. For an application of set-membership state estimation methods for model falsification see [18]. Zonotopic set-membership estimation is also used recently for robust tube-based output feedback model predictive control [12].

This paper considers the problem of set-membership state estimation for discrete time piecewise affine (PWA) systems.

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PWA systems can approximate nonlinear systems effectively. Moreover, many nonlinear systems that contain PWA components such as deadzone, saturation, hysteresis, etc can be modeled efficiently as PWA systems. The PWA modeling framework is an attractive modeling framework for such systems [9], [7].

PWA systems has attracted a lot of attention in the last decade and many synthesis and control problems of them is addressed in the literature. To the best of authors knowledge, the problem of set-membership state estimation of PWA systems has not been paid enough attention. In [17] a methods based on polyhedral sets is proposed. As explained before, polyhedral sets suffer form high computational complexity that is even intensified for the case of PWA systems. In this paper, we use zonotopic sets to deal with the problem of computational complexity. Because in piecewise affine system, the state space is partitioned into a number of polyhedral sets, at each iteration the intersection of the zonotopes containing a set-valued estimation of the states with each of the polyhedral partitions must be computed. We use an analytic method to compute the intersection as a zonotope and minimize the size of the intersection.

This paper is organized as follows. First, in section II, preliminaries and basic definitions are given. In section III we introduce PWA systems and formulate the problem. In section IV, the general algorithm for set-membership state estimation is given. Then in section V we explain how the algorithm is implemented using zonotopes. Finally the paper concludes in Section VII with conclusion.

## II. PRELIMINARIES

Given two sets  $\mathcal{X} \in \mathbb{R}^n$  and  $\mathcal{Y} \in \mathbb{R}^n$ , the Minkowski sum of them is defined as  $\mathcal{X} \oplus \mathcal{Y} = \{x + y | x \in \mathcal{X}, y \in \mathcal{Y}\}$ . A strip  $S$  is defined by the set  $S = \{x \in \mathbb{R}^n | |cx - d| \leq \sigma\}$ , where  $c, \in \mathbb{R}^{1 \times n}$  and  $d, \sigma \in \mathbb{R}$ . A convex polytope  $P$  is the convex combination of its vertices. The polytope  $P$  with  $r$  vertices  $v^i \in \mathbb{R}^n$  is the set:

$$P = \left\{ \sum_{i=1}^r \alpha^i v^i \mid v^i \in \mathbb{R}^n, \alpha^i \in \mathbb{R}, \alpha^i \geq 0, \sum_{i=1}^r \alpha^i = 1 \right\}. \quad (1)$$

$P$  can also be represented by the nonempty intersection of a finite set of half-spaces. Zonotopes are a special class of convex polytopes. A zonotope of order  $m$  in  $\mathbb{R}^n$  is an affine image of a  $m$ -dimensional unitary box  $\mathbb{B}^m$  in  $\mathbb{R}^n$ . Given the vector  $p \in \mathbb{R}^n$ , and the matrix  $H \in \mathbb{R}^{n \times m}$ , then the set

$$p \oplus H\mathbb{B}^m = \{p + Hz \mid z \in \mathbb{B}^m\} \quad (2)$$

is a zonotope which is the affine image of  $\mathbb{B}^m$  defined by  $p, H$ . Here,  $p$  is the center of zonotope. A zonotope can also be considered as the Minkowski sum of a finite number of line segments. In this case the zonotope is represented by:

$$Z = \{z \in \mathbb{R}^n \mid z = c + \sum_{i=1}^p x_i g_i, -1 \leq x_i \leq 1\}. \quad (3)$$

Here,  $c$  is the center of zonotope and  $g_i$ 's are called generators. Therefore, the zonotope  $Z = p \oplus H\mathbb{B}^m$  is actually the Minkowski sum of the line segments defined by columns

of  $H$  centered on  $p$ . Zonotopes are interesting objects in the field of computational geometry because they are closed under the Minkowski sum and under linear transformation.

## III. PIECEWISE AFFINE SYSTEMS AND PROBLEM FORMULATION

We consider a PWL discrete time system of the following form:

$$x(k+1) = f(x(k), u(k), w(k)), \quad (4)$$

$$y(k) = g(x(k), v(k)) \quad (5)$$

where  $x(k) \in \mathbb{R}^n$  is the state,  $u(k) \in \mathbb{R}^m$  is the control input,  $w(k) \in \mathbb{R}^n$  is the disturbance input,  $y(k) \in \mathbb{R}^p$  is the measured output, and  $v(k) \in \mathbb{R}^p$  is noise on the measurement. The noise and disturbance are assumed to be unknown but bounded in a given compact set i.e  $v(k) \in \mathcal{V}$  and  $w(k) \in \mathcal{W}$ . The functions  $f(\cdot)$  and  $g(\cdot)$  are piecewise affine functions defined as:

$$\begin{cases} f(x, u, w) = A_i x + B_i u + w, \\ g(x, v) = C_i x + v, \end{cases} \quad \text{for } x \in \mathcal{R}_i, i \in \mathcal{I}, \quad (6)$$

where  $A_i, B_i, C_i$  are constant real matrices with appropriate dimensions.  $\{\mathcal{R}_i\}_{i=1}^s \subseteq \mathbb{R}^n$  denotes a partition of the state space into a number of polyhedral regions  $\mathcal{R}_i, i \in \mathcal{I} = \{1, \dots, s\}$ . Each polyhedral region is given by  $\mathcal{R}_i = \{x \mid H_i x \leq K_i\}$ . We also assume that the initial state is given as a bounded compact set i.e.  $x(0) \in \mathcal{X}_0$ . The problem that we address in this paper is the follwoign. Given the initial state  $x(0) \in \mathcal{X}_0$ , given the input sequence  $\mathbf{u}_k = \{u(0), u(1), \dots, u(k-1)\}$ , the observation sequence  $\mathbf{y}_k = \{y(0), u(1), \dots, y(k)\}$ , find a set  $\mathcal{X}^c(k)$  such that it is guaranteed that the true state  $x(k)$  lies in this set. Let  $\mathbf{w}_k$  denote the disturbance sequence  $\{w(0), w(1), \dots, w(k-1)\}$  and  $\mathbf{v}_k$  denote the measurement noise sequence  $\{v(0), v(1), \dots, v(k-1)\}$ .

**Definition 1:** A state  $x^c$  is said to be consistent with the initial state set  $\mathcal{X}_0$ , the input sequence  $\mathbf{u}_k = \{u(0), u(1), \dots, u(k-1)\}$ , the observation sequence  $\mathbf{y}_k = \{y(0), u(1), \dots, y(k)\}$  if there exist a disturbance sequence  $\mathbf{w}_k = \{w(0), w(1), \dots, w(k-1)\}$ ,  $w(k) \in \mathcal{W}$  and a measurement noise sequence  $\mathbf{v}_k = \{v(0), v(1), \dots, v(k)\}$ ,  $v(k) \in \mathcal{V}$  such that  $x^c = x(k)$  and  $y(k) = g(x^c, v(k))$ , where  $x(l) = f(x(l-1), u(l-1), w(l-1))$  for  $l \in \{1, \dots, k\}$  and  $y(l) = g(x(l), v(l))$  for  $l \in \{0, \dots, k\}$ .

The problem of set-membership estimation is to find the set of all the states at time  $k$ ,  $\mathcal{X}^c(k)$ , that are consistent with the initial state, the input and the output sequence  $\mathbf{u}_k, \mathbf{y}_k$  i.e.

$$\begin{aligned} \mathcal{X}^c(k) &= \{x(k) \mid x(l) = f(x(l-1), u(l-1), w(l-1)), \\ &\quad l \in \{1, \dots, k\}, y(\xi) = g(x(\xi), v(\xi)), \\ &\quad x(0) \in \mathcal{X}_0, w(l) \in \mathcal{W}, v(\xi) \in \mathcal{V}, \xi \in \{0, \dots, k\}\}. \end{aligned} \quad (7)$$

## IV. SET-MEMBERSHIP STATE ESTIMATION FOR PIECEWISE AFFINE SYSTEMS

Most of the set-membership algorithms use a recursive method to find an over-approximation of the consistent set  $\mathcal{X}^c(k)$ . Computation of  $\mathcal{X}^c(k)$  consist of two steps: a prediction step and a correction step. At the prediction step,

TABLE I

THE GENERAL ALGORITHM FOR SET-MEMBERSHIP STATE ESTIMATION

**Algorithm 1**


---

**Given**  $f, g, \mathcal{X}_0, \mathcal{V}, \mathcal{W}$   
 $k \leftarrow 0, \mathcal{X}^c(k) \leftarrow \mathcal{X}_0$   
**While** (There is data)  
 $k \leftarrow k + 1$   
 Given  $u(k)$ , find the prediction set:  
 $\mathcal{X}^p(k) \leftarrow \{f(x, u, w) : x \in \mathcal{X}^c(k-1), u = u(k-1), w \in \mathcal{W}\}.$   
 Given  $y(k)$ , find  $\mathcal{X}^y(k)$ :  
 $\mathcal{X}^y(k) \leftarrow \{x \in \mathbb{R}^n : \exists v \in \mathcal{V}, g(x, v) = y(k)\}.$   
 $\mathcal{X}^c(k) \leftarrow \mathcal{X}^p(k) \cap \mathcal{X}^y(k)$   
**end**

---

having  $\mathcal{X}^c(k-1)$ , the predicted set  $\mathcal{X}^p(k)$  is defined as the set of states that are reachable by the system in one step given the input:

$$\mathcal{X}^p(k) = \{f(x, u, w) | x \in \mathcal{X}^c(k-1), u = u(k-1), w \in \mathcal{W}\}. \quad (8)$$

The predicted set is one-step ahead prediction of  $\mathcal{X}^c(k-1)$ . This set is then corrected using the information provided by the current measurement  $y(k)$ . Let us define  $\mathcal{X}^y(k)$  as the set of all states that are consistent with  $y(k)$ :

$$\mathcal{X}^y(k) = \{x \in \mathbb{R}^n : \exists v \in \mathcal{V} \text{ such that } g(x, v) = y(k)\}. \quad (9)$$

Then, the corrected set is defined as:

$$\mathcal{X}^c(k) = \mathcal{X}^p(k) \cap \mathcal{X}^y(k). \quad (10)$$

The overall algorithm is given in table I. In the case of PWA systems, to perform the prediction step, intersection of the consistent set  $\mathcal{X}^c(k)$  with each region  $\mathcal{R}_i, i \in \mathcal{I}$  must be computed, and then the prediction is computed based on the local dynamic of each region for each intersection that is not empty. Assume that the consistent set at time  $k-1$  is given as a union of  $J_k$  sets i.e  $\mathcal{X}^c(k-1) = \bigcup_{j=1}^{J_k} \mathcal{X}_j^c(k-1)$ . Then, for each  $\mathcal{X}_j^c(k-1)$ , its intersection with  $\mathcal{R}_i$  is found:

$$\mathcal{X}_{j,i}^c(k-1) = \mathcal{X}_j^c(k-1) \cap \mathcal{R}_i. \quad (11)$$

Then, the prediction set is calculated as:

$$\mathcal{X}^p(k-1) = \bigcup_{j \in J_k, i \in \mathcal{I}} A_i \mathcal{X}_{j,i}^c(k-1) + B_i u(k) \oplus \mathcal{W} \quad (12)$$

Also to find  $\mathcal{X}^y(k)$ , we find the consistent states with the output for each region based on its corresponding output matrix:

$$\mathcal{X}^y(k) = \bigcup_{i \in \mathcal{I}} \{x | C_i x \oplus (-\mathcal{V}) = y(k)\} \cap \mathcal{R}_i. \quad (13)$$

Then, the corrected set is given by:

$$\mathcal{X}^c(k) = \left( \bigcup_{j \in J_k, i \in \mathcal{I}} A_i \mathcal{X}_{j,i}^c(k-1) + B_i u(k) \oplus \mathcal{W} \right) \cap \left( \bigcup_{i \in \mathcal{I}} \{x | C_i x \oplus (-\mathcal{V}) = y(k)\} \cap \mathcal{R}_i \right) \quad (14)$$

TABLE II

SET-MEMBERSHIP STATE ESTIMATION FOR PWA SYSTEMS

**Algorithm 2**


---

**Given**  $A_i, B_i, C_i, \mathcal{R}_i, s, \mathcal{X}_0, \mathcal{V}, \mathcal{W}$   
**Output** consistent sets  $\mathcal{X}^c(k)$  for each iteration  
 $k \leftarrow 0, N^i \leftarrow 1, \mathcal{X}_{1,1}^p(k) \leftarrow \mathcal{X}_0$   
**While** (There is data)  
**For**  $i = 1$  **to**  $s$   
 $N_t^i \leftarrow 0$   
**For**  $j = 1$  **to**  $N^i$   
 $XR_{j,i}(k) \leftarrow \mathcal{X}_{j,i}^p(k) \cap \mathcal{R}_i$   
**If**  $XR_{j,i}(k) \neq \emptyset$   
 $\mathcal{X}_i^y(k) \leftarrow \{x | C_i x \oplus \mathcal{V} = y(k)\}$   
 $XO_{j,i}(k) \leftarrow \mathcal{X}_i^y(k) \cap XR_{j,i}$   
**If**  $XO_{j,i}(k) \neq \emptyset$   
 $N_t^i \leftarrow N_t^i + 1$   
 $\mathcal{X}_{N_t^i, i}^c(k) \leftarrow XO_{j,i}(k)$   
 $\mathcal{X}_{N_t^i, i}^p(k+1) \leftarrow A_i \mathcal{X}_{N_t^i, i}^c(k) + B_i u(k) \oplus \mathcal{W}$   
**End If**  
**End If**  
 $N^i \leftarrow N_t^i$   
**End**  
 $\mathcal{X}^c(k) = \bigcup_{i=1}^s \bigcup_{j=1}^{N^i} \left( \mathcal{X}_{j,i}^c(k) \right)$   
 $k \leftarrow k + 1$   
**End**

---

The overall algorithm for set-membership state estimation of PWA systems is given in table II. As it can be seen from the algorithm, computation for each subsystem is independent at each iteration. Therefore, these computations can be easily parallelized.

## V. IMPLEMENTATION USING ZONOTOPES

The algorithm 2 is given in the general form. Computational complexity of the algorithm depends on the computational complexity of the specific set-representation that is used to implement the algorithm with respect to the operations that must be performed. These operations are: affine transformation, Minkowski sum, intersection with a strip and intersection with polyhedral sets. Zonotopes are closed under affine transformation and Minkowski sum and they offer low time and memory complexity. For a system with dimension  $n$ , computational complexity for linear transformation of a zonotope is  $\mathcal{O}(n^3)$  and for the Minkowski sum of two zonotopes is  $\mathcal{O}(n)$ . For calculating the intersection of the output consistent set with the predicted we use the segment minimization method proposed in [1] which is a computationally efficient method that over-approximates the intersection as a zonotope. To calculate intersection of a zonotope with each polyhedral region, we modify and adapt the idea of the segment minimization method. Therefore, the overall algorithm would have a low computational complexity. In the following, we explain how to perform each of the

required operations using zonotopes. The set-operations that must be performed are:

- Minkowski sum of two zonotopes
- Linear mapping of a zonotope
- Calculating the intersection of a strip and a zonotope
- Calculating the intersection of a zonotope and a polyhedra

#### A. Minkowski sum of two zonotopes

We use the following property to compute the Minkowski sum of two zonotopes:

*Property 1:* Given two zonotopes  $Z_1 = p_1 + H_1 \mathbf{B}_1^m$  and  $Z_2 = p_2 + H_2 \mathbf{B}_2^m$ , then the Minkowski sum of them is also a zonotope and we have:

$$Z_1 \oplus Z_2 = (p_1 + p_2) \oplus [H_1 \ H_2] \mathbf{B}^{m_1+m_2}. \quad (15)$$

In other words, to obtain the Minkowski sum of two zonotopes, one needs to add their centers and concatenate their generators.

#### B. Linear mapping of a zonotope

The following property is used to find a linear image of a zonotope.

*Property 2:* Given a zonotope  $Z = p + H\mathbf{B}^m$  and a linear map  $L$ , the image of  $Z$  by  $L$  is a zonotope given by:

$$LZ = Lp \oplus [LH] \mathbf{B}^m. \quad (16)$$

In other words, we just need to transform the generators by the linear map.

#### C. Computing the intersection of a zonotope and a strip

One step of the algorithm 2, is to compute the intersection of the set of states that are consistent with the current measurement with the predicted set. Assuming the predicted set is given as a union of zonotopes, this operation boils down to computing the intersection of a zonotope and a finite number of strips. The set of consistent states with the measurement,  $\mathcal{X}_i^y(k)$ , can be considered as the intersection of  $p$  strips where  $p$  is the number of rows in the output matrix  $C_i$ . Assume that the set  $\mathcal{V}$  is given as a hyper-rectangle i.e.  $\mathcal{V} = [-v_1, v_1] \times \dots \times [-v_p, v_p]$ . Given a measurement  $y(k)$ , the set of states that are consistent with it is given by:

$$\mathcal{X}_i^y(k) = \{x \in \mathbb{R}^n \mid |C_i x(k) - y(k)| \leq v\}, \quad (17)$$

where  $v = [v_1, \dots, v_p]^T$ . This set can be viewed as the intersection of  $p$  strips:

$$\mathcal{X}_i^y(k) = \bigcap_{l=1}^p \mathcal{X}_i^{y_l}(k), \quad (18)$$

where  $\mathcal{X}_i^{y_l}(k)$  denotes the strip which contains the set of states consistent with the  $l$ 'th element of the measurement  $y_l(k)$  which is:

$$\mathcal{X}_i^{y_l}(k) = \{x \in \mathbb{R}^n \mid |c_i^l x - y_l(k)| \leq v_l\}, \quad (19)$$

where  $c_i^l$  is the  $l$ 'th row of  $C_i$ .

Consequently, computing the intersection of  $\mathcal{X}_i^y(k)$  and a zonotope amounts to computing the intersection of a

zonotope and a strip sequentially such that at each iteration the intersection is over-approximated by a zonotope. At each iteration before proceeding with the computation, we check if the zonotope and the corresponding strip intersect.

**-Checking consistency of a zonotope and a strip:** To check if a zonotope and a strip intersect we find the support strip of the zonotope in the direction of  $\mathbf{c}$ . A support strip of a zonotope for given a direction, is a strip such that the zonotope is inside the strip and both the hyperplanes defining the strip touch the zonotope from each side.

*Definition 2:* [22] Given a zonotope  $Z = p \oplus H\mathbf{B}^m$  and a strip  $S = \{x \in \mathbb{R}^n \mid |cx - d| \leq \sigma\}$ , the zonotope support strip is defined by

$$F_S = \{x \in \mathbb{R}^n \mid q_d \leq cx \leq q_u\}, \quad (20)$$

where  $q_u$  and  $q_d$  are defined as:

$$q_u = \max_{x \in Z} cx, \quad (21)$$

$$q_d = \min_{x \in Z} cx, \quad (22)$$

which are calculated by:

$$q_u = \mathbf{c}p + \|H^T c\|_1, \quad (23)$$

$$q_d = \mathbf{c}p - \|H^T c\|_1. \quad (24)$$

where  $\|\cdot\|_1$  is the 1-norm of a vector. Then,  $S \cap Z = \emptyset$  if and only if:

$$q_u < \frac{d}{\sigma} - 1 \text{ or } q_d > \frac{d}{\sigma} + 1. \quad (25)$$

**-Intersection of a zonotope and a strip:** Given a zonotope and a strip, the following property gives a family of zonotopes parameterized by the vector  $\lambda$  that over-approximates the intersection of the zonotope and the strip.

*Property 3:* [1] Given the zonotope  $Z = p \oplus H\mathbf{B}^r \subset \mathbb{R}^n$ , the strip  $S = \{x \in \mathbb{R}^n \mid |cx - d| \leq \sigma\}$  and the vector  $\lambda \in \mathbb{R}^n$ , define:

$$\hat{p}(\lambda) = p + \lambda(d - cp), \quad (26)$$

$$\hat{H}(\lambda) = [(I - \lambda c)H \quad \sigma\lambda]. \quad (27)$$

Then  $S \cap Z \subseteq \hat{p}(\lambda) + \hat{H}(\lambda)\mathbf{B}^{r+1}$ .

The above over-approximation might not be a good approximation. To find an appropriate over-approximation,  $\lambda$  must be chosen such that an approximation criterion is minimized. In [1] two approaches are proposed. The first approach is a segment minimization approach which provides a low computational complexity. The second approach which provides a better approximation is a volume-minimizing approach. The second approach requires solving a convex optimization problem at each iteration. Here, we choose the first approach. In the first approach the segments of zonotopes are minimized by minimizing the Frobenius norm of  $\hat{H}(\lambda)$ . The  $\lambda$  that minimizes the Frobenius norm of  $\hat{H}(\lambda)$  is given by:

$$\lambda^* = \frac{HH^T c^T}{cHH^T c^T + \sigma^2}. \quad (28)$$

The advantage of this approach is its computational simplicity.



**-Intersection of a zonotope and a polyhedron:** A polyhedron  $P = \{x | Ex \leq F\}$  is actually the intersection of a finite number of half-spaces, i.e

$$P = \bigcap_{i=1}^r \mathcal{H}^i, \quad \mathcal{H}^i = \{x | E^i \leq F^i\}, \quad (29)$$

where  $E^i$  and  $F^i$  denote the  $i$ 'th row of the matrices  $E$  and  $F$  respectively. Consequently, to find the intersection of a zonotope  $Z = p \oplus H\mathbf{B}^m$  and the polyhedron  $P$ , we need to find the intersection of a zonotope and a half-space. If we over-approximate this intersection as a zonotope, then we can compute  $Z \cap P$  by sequential computation of intersection of a zonotope and  $H^i$ 's. This is shown in Algorithm 3 where  $\text{OV\_INT\_ZH}(Z, \mathcal{H})$  is a subroutine that over-approximates the intersection of the zonotope  $Z$  and the half-space  $\mathcal{H}$  by a zonotope.

To compute the intersection of a zonotope  $Z$  and a half-space  $\mathcal{H} = \{x | \eta x \leq \gamma\}$ , we first find a tight supporting strip for the zonotope  $Z$  given the direction  $\eta$  denoted as  $S_Z$ :

$$S_Z = \{x \in \mathbb{R}^n | q_d \leq \eta x \leq q_u\}, \quad (30)$$

Three cases are possible. The first one is that  $Z$  and  $\mathcal{H}$  does not intersect. In this case, we have  $q_l > \gamma$ . The second case is when  $Z \subset \mathcal{H}$  which is equal to  $q_u \leq \gamma$ . In this case, the intersection is  $Z$  itself and no further calculation is required. The last case is when  $q_u > \gamma$  but  $q_d \leq \gamma$ . In this case, we have to over-approximate the intersection as a zonotope. The set  $Z \cap \mathcal{H}$  is actually bounded in the direction  $\eta$  by the hyperplane  $\mathcal{H}$ . Moreover, in the direction  $-\eta$  it is bounded by the hyperplane defined as  $\{x \in \mathbb{R}^n | -\eta x \leq q_d\}$ . This means that the we have:

$$x \in Z \cap \mathcal{H} \rightarrow q_d \leq \eta x \leq \gamma, \quad (31)$$

Therefore, the tight supporting strip for the intersection, given the direction  $\eta$  is:

$$S_{Z \cap \mathcal{H}} = \{x \in \mathbb{R}^n | q_d \leq \eta x \leq \gamma\} \quad (32)$$

. This is a strip with  $\sigma = \frac{\gamma - q_d}{2}$  and  $d = \frac{q_d + \gamma}{2}$ . Now, the problem is to find the intersection of  $Z$  and the strip  $S_{Z \cap \mathcal{H}}$ . From the last subsection we know that this intersection can be found using the segment minimization method. The over-approximation is, therefore, defined by  $Z_{int} = \hat{p} \oplus \hat{H}\mathbf{B}^{m+1}$  where:

$$\hat{p} = p + \lambda^*(d - \eta p) \quad (33)$$

$$\hat{H} = [(I - \lambda^*\eta)H \quad \sigma\lambda^*] \quad (34)$$

where  $\lambda^* = \frac{HH^T\eta^T}{\eta HH^T\eta^T + \sigma^2}$ . The overall algorithm is given in Table III.

## VI. EXAMPLE

To illustrate the proposed method we consider the following PWA system:

$$x(k+1) = \begin{cases} A_1x(k) + f_1 + w & \text{if } x(k) < 1.5 \\ A_2x(k) + f_2 + w & \text{if } x(k) \geq 1.5 \end{cases}, \quad (35)$$

$$y(k) = \begin{cases} C_1x(k) + v & \text{if } x(k) < 1.5 \\ C_2x(k) + v & \text{if } x(k) \geq 1.5 \end{cases}, \quad (36)$$

TABLE III

THE PROPOSED ALGORITHM FOR ZONOTOPIC OVER-APPROXIMATION OF INTERSECTION OF A ZONOTOPE AND A POLYHEDRON

### Algorithm 3

**Given:** zonotope  $Z$  and Polyhedron  $P$

**Output:** a zonotope  $Z_{int}$  over-approximating  $Z \cap P$

$Z_{int} = Z$

**For**  $i = 1$  to  $r$

$Z_{int} \leftarrow \text{OV\_INT\_ZH}(Z_{int}, H^i)$

**end**

**Subroutine:**  $\text{OV\_INT\_ZH}(Z, \eta)$

**Given:** Zonotope  $Z$  and vector  $\eta$

**Output:** a zonotope  $Z_{int}$  over-approximating  $Z \cap \mathcal{H}$

with  $\mathcal{H} = \{x | \eta x \leq \gamma\}$

$q_u = \eta p + \|H^T\eta\|_1$

$q_d = \eta p - \|H^T\eta\|_1$

**If**  $q_d \leq \gamma$

**If**  $q_u \leq \gamma$

$Z_{int} = Z$

**Else**

$\sigma = \frac{\gamma - q_d}{2}$

$d = \frac{q_d + \gamma}{2}$

$\lambda^* = \frac{HH^T\eta^T}{\eta HH^T\eta^T + \sigma^2}$

$\hat{p} = p + \lambda^*(d - \eta p)$

$\hat{H} = [(I - \lambda^*\eta)H \quad \sigma\lambda^*]$

$Z_{int} = \hat{p} \oplus \hat{H}\mathbf{B}^{m+1}$  **Else**

$Z_{int} = \emptyset$

**End If**

**End If**

**End**

where

$$A_1 = \begin{bmatrix} 0.7969 & -0.2247 \\ 0.1798 & 0.9767 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0.4969 & -0.2247 \\ 0.0798 & 0.9767 \end{bmatrix}$$

$$f_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad f_2 = \begin{bmatrix} 0.3 \\ 0.1 \end{bmatrix}$$

$$C_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0.5 & 0 \end{bmatrix}.$$

It is assumed that noise and disturbance are in the following sets:

$$\mathcal{W} = \{w \in \mathbb{R}^2 | \|w\|_\infty \leq 0.05\}$$

$$\mathcal{V} = \{v \in \mathbb{R} | -0.05 \leq v \leq 0.05\}$$

Also, the initial state is assumed to be in the set:

$$\mathcal{X}_0 = (5, 3) \oplus 5 \times \mathbf{B}^2$$

. The initial state is chosen to be  $(8, -1)$ . The estimation result is shown in Figure 1. As it can be seen, the true state is always inside the estimated sets. The initial set and the evolution of the predicted sets are depicted in Figure 2. The blue set shows the intersection with the first region,  $x < 1$ , and the red sets shows the intersection with the second region  $x \geq 1$ . By comparing the two figures, we can see that the estimated sets, are the intersection of the output consistent sets and the predicted sets.

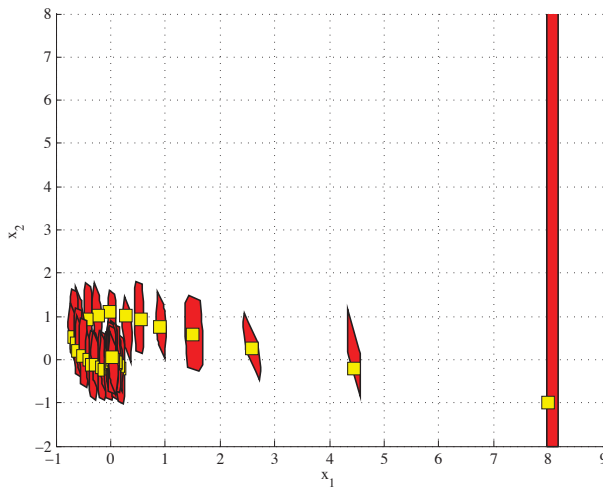


Fig. 1. The estimated sets ( $\mathcal{X}^p(k) \cap \mathcal{X}^y(k)$ ), the truer state (yellow squares)

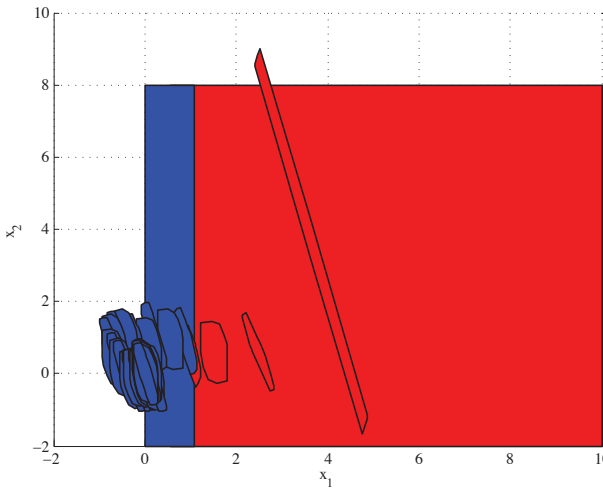


Fig. 2. The initial set and predicted sets and their intersection with each regions (blue: intersection with region 1, red: intersection with region 2)

## VII. CONCLUSIONS

In this paper a computationally efficient method for state estimation of discrete time piecewise affine systems with bounded noise and disturbance is proposed. We used zonotopes for over-approximation of the estimated sets. The disturbance and noise are explicitly taken into account such that it is guaranteed that the set-valued estimation contains the true state of the system. To compute the intersection of the zonotopic sets, with each polyhedral regions in the PWA systems, we modified the segment minimization method for this problem. Consequently, the intersection is over-approximated as a zonotope using an analytic expression. Therefore, the overall algorithm is based on zonotopic set-representation which yields a computationally efficient method. A numerical example is used to demonstrate the method.

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